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Solution to the Nonlinear Boltzmann Equation for Maxwell Models for Nonisotropic Initial Conditions

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The known solution to the spatially homogeneous nonlinear Boltzmann equation for Maxwell models in a series of Laguerre polynomials is extended to include nonisotropic initial conditions. Existence proofs for a class of solutions are supplied. The equations for the generalized (nonisotropic Laguerre) moments are derived in explicit form for two- and three-dimensional models. Further it is shown that the ordinary moments satisfy the same set of equations as the (Hermite) polynomial moments.

KEY WORDS: Nonlinear Boltzmann equation; nonisotropic initial conditions; exact solutions; moment equations.

INTRODUCTION

In recent years there has been a considerable revival of interest in the theory of the nonlinear Boltzmann equation. This equation describes the nonequilibrium properties of dilute monatonic gasses. See Ref. 1 and its references.

One of the major results applies to the so-called Maxwell models (these correspond to a molecular scattering cross section proportional to $|\mathbf{v}-\mathbf{v}_1|^{-1}$, where $\mathbf{v}$ and $\mathbf{v}_1$ are the velocities of two colliding particles). It is the discovery of the general solution[(2)] for the spatially homogeneous case and isotropic initial conditions—within a certain Hilbert space of functions—as an expansion in terms of the eigenfunctions of the corresponding linearized equation. The coefficients in this expansion are functions of time to be determined from a recursively soluble set of nonlinear moment equations.

Proofs that for a class of initial conditions the expansion converges to a solution of the Boltzmann equation have been given later.[(17,18)]

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In this paper we generalize this result to include nonisotropic initial conditions. In short, we shall (i) show that certain (nonisotropic Laguerre) moments can be defined, satisfying a recursively soluble set of coupled nonlinear equations, (ii) derive—in an easy manner—the explicit form of these equations for models in two and three dimensions, and (iii) show that the velocity distribution function can be expanded in terms of eigenfunctions of the linearized equations, the coefficients of which are precisely the above-mentioned moments. We use a moment method to solve the nonlinear Boltzmann equation in combination with Fourier transformation as first applied by Bobylev.[5] Moment methods have been used before in the nonisotropic case by Ikenberry and Truesdell[4] and Grad[3] (see discussion).

Kumar[15] has given the expansion of the collision integral in terms of the same basis functions as those used here for general cross section in a rather elegant way. The moment equations that follow from his result [his Eq. (100)] are a generalization of our result (ii). However, the result of the present paper, which deals only with Maxwell models, is much more explicit as it contains only Clebsch–Gordan coefficients rather than the more complicated and lesser known Talmi coefficients.

In Section 1 we introduce the basic concepts. After that we shall solve the two-dimensional case (Section 2) and the three-dimensional case (Section 3) in the sense indicated above. We treat the two-dimensional case first for pedagogical reasons. After briefly discussing the ordinary moments and the related (Hermite) polynomial moments (Section 4) we conclude with a discussion of the result.

1. BASIC CONCEPTS

We consider the spatially homogeneous nonlinear Boltzmann equation in d dimensions ($d = 2, 3$):[1]

$$\frac{\partial f}{\partial t}(\mathbf{v}, t) = \int d\mathbf{v}_1\, d\hat{n}\; gI(\chi, g)\left[f(\mathbf{v}', t) f(\mathbf{v}_1', t) - f(\mathbf{v}, t) f(\mathbf{v}_1, t)\right] \tag{1.1}$$

for the one particle velocity distribution function $f(\mathbf{v}, t)$. The postcollisional (primed) velocities are given in terms of the incoming velocities by the dynamics:

$$\begin{aligned} \mathbf{v}' &= \tfrac{1}{2}(\mathbf{v} + \mathbf{v}_1) + \tfrac{1}{2}\hat{n}g \\ \mathbf{v}_1' &= \tfrac{1}{2}(\mathbf{v} + \mathbf{v}_1) - \tfrac{1}{2}\hat{n}g \end{aligned} \tag{1.2}$$

where $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$ is the relative velocity of the colliding particles, $g \equiv |\mathbf{g}|$ and $\hat{n}$ a d-dimensional unit vector, parametrizing the collision. χ is the scattering angle such that $\cos\chi = \hat{n} \cdot \hat{g}$ (hats denote unit vectors throughout this paper). The differential cross section $I(\chi, g)$ has the following form for Maxwell models:[1]

$$I(\chi, g) = \sigma(\cos\chi)/g \tag{1.3}$$

A special case of this is embodied by the so-called Maxwell molecules, molecules interacting through a repelling $r^{2(1-d)}$ potential (in d-dimensions), giving rise to a cross section of the above form with $\sigma(\cos\chi)$ some well-defined though complicated functions[16]. More general models can be defined by choosing the function $\sigma(\cos\chi)$ conveniently. Then the cross section is in general not derivable from a potential.

The special form of the cross section (1.3) yields a great simplification of the nonlinear Boltzmann equation. It has been noted by Bobylev[5] that the Fourier-transformed equation—which is in general just as difficult as the original equation—is much simpler for this case. The number of integrations in the collision term reduces from $2d-1$ to $d-1$. Some important symmetry properties—now called Bobylev symmetries[1]—can be simply read off in this representation.

The Fourier transform—or characteristic function—$\Phi(\mathbf{k}, t)$ is defined as follows:

$$\Phi(\mathbf{k}, t) = \langle e^{i\mathbf{k}\cdot\mathbf{v}}\rangle \equiv \int e^{i\mathbf{k}\cdot\mathbf{v}} f(\mathbf{v}, t)\, d\mathbf{v} \tag{1.4}$$

[Here and in the sequel angle brackets will denote averaging with respect to $f(\mathbf{v}, t)$, as indicated]. For Maxwell models, with a cross section given by (1.3), it satisfies the following equation, derived by Bobylev[5]

$$\frac{\partial\Phi(\mathbf{k}, t)}{\partial t} = \int d\hat{n}\, \sigma(\hat{k}\cdot\hat{n})\left\{\Phi\left[\tfrac{1}{2}k(\hat{k}+\hat{n}), t\right]\Phi\left[\tfrac{1}{2}k(\hat{k}-\hat{n}), t\right]\right.$$
$$\left. - \Phi(\mathbf{k}, t)\Phi(\mathbf{0}, t)\right\} \tag{1.5}$$

Note that only the even part $\bar{\sigma}(\hat{k}\cdot\hat{n}) \equiv \frac{1}{2}[\sigma(\hat{k}\cdot\hat{n}) + \sigma(-\hat{k}\cdot\hat{n})]$ of the cross section contributes to the integral.

The small-$\mathbf{k}$ behavior of $\Phi(\mathbf{k}, t)$ must be compatible with the conservation laws for particle number, total momentum, and energy. By a proper choice of units and frame of reference we can write these as

$$\begin{aligned}\Phi(\mathbf{0}, t) &= \langle 1\rangle = 1\\ \nabla_{\mathbf{k}}\Phi(\mathbf{k}, t)|_{\mathbf{k}=\mathbf{0}} &= i\langle\mathbf{v}\rangle = \mathbf{0}\\ \nabla^2_{\mathbf{k}}\Phi(\mathbf{k}, t)|_{\mathbf{k}=\mathbf{0}} &= -\langle\mathbf{v}^2\rangle = -d\end{aligned} \tag{1.6}$$

For isotropic initial conditions Eq. (1.5) has been discussed *in extenso*. The Laguerre series solution can be derived from it by expanding $\Phi(\mathbf{k}, t)$ in a Taylor series around $\mathbf{k}=\mathbf{0}$.[2] Here we apply a similar method for nonisotropic initial conditions, first for the much simpler two-dimensional case. For the physically more relevant three-dimensional case the calculations are more complicated, though the results of both cases are essentially the same.

2. SOLUTION TO THE TWO-DIMENSIONAL MODEL

In two dimensions the Fourier-transformed Boltzmann equation reads

$$\frac{\partial\Phi(\mathbf{k},t)}{\partial t}=\int_0^{2\pi}d\phi\,\bar{\sigma}[\cos(\phi-\alpha)]\times\left\{\Phi[\tfrac{1}{2}k(\hat{k}+\hat{n}),t]\Phi[\tfrac{1}{2}k(\hat{k}-\hat{n}),t]-\Phi(\mathbf{k},t)\right\}\tag{2.1}$$

with α and ϕ defined as the polar angles belonging to $\mathbf{k}$ and $\hat{n}$,

$$\begin{aligned}\mathbf{k}&=k(\cos\alpha,\sin\alpha),\qquad 0\leqslant\alpha\leqslant 2\pi\\ \hat{n}&=(\cos\phi,\sin\phi),\qquad 0\leqslant\phi\leqslant 2\pi\end{aligned}\tag{2.2}$$

If the moments of the velocity distribution function exist we can expand the characteristic function in a series (which we assume to converge):

$$\Phi(\mathbf{k},t)=\exp(-\tfrac{1}{2}k^2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n}{}' C_{nm}(t)(ik)^n\exp(im\alpha)\tag{2.3}$$

In Appendix A it is derived that the summation is over values for which $n+m$ is even [so we define $C_{nm}(t)=0$ if $n+m$ is odd and indicate this by the prime] and that the coefficients $C_{nm}(t)$ are generalized moments of the velocity distribution function:

$$\begin{aligned}C_{nm}(t)&=\frac{(\frac{1}{2})^{(n+|m|)/2}(-)^{(n-|m|)/2}}{[(n+|m|)/2]!}\langle\mathcal{L}^{(|m|)}_{(n-|m|)/2}(\tfrac{1}{2}v^2)v^{|m|}e^{-im\beta}\rangle\\ \mathbf{v}&\equiv v(\cos\beta,\sin\beta)\end{aligned}\tag{2.4}$$

which have a very simple asymptotic (large t) form:

$$C_{nm}(\infty)=\delta_{n0}\delta_{m0}\tag{2.5}$$

as follows from the fact that the velocity distribution function approaches the Maxwellian, the Fourier transform of which is given by $\Phi(\mathbf{k},\infty)=\exp(-\frac{1}{2}k^2)$. The symbol $\mathcal{L}^{(\alpha)}_n(x)$ denotes the associated Laguerre polynomial. We shall call the above moments nonisotropic Laguerre moments.

[2] The assumption that the Taylor series exists restricts the class of solutions to a certain function space. However, (1.5) has also solutions outside this space.[6,7]

The functions

$$E_{nm}(\mathbf{v}) = \mathbf{v}^{|m|}\mathfrak{L}_{(n-|m|)/2}^{(|m|)}\left(\tfrac{1}{2}v^2\right)e^{im\beta}$$
$$-n \leqslant m \leqslant n, \qquad n+m \quad \text{even} \tag{2.6}$$

form a complete set in the Hilbert space of functions $f(v)$ for which $\|f\|^2 = \int d\mathbf{v}\exp(\frac{1}{2}v^2)|f(\mathbf{v})|^2$ is finite. They are known to be eigenfunctions of the Hamiltonian of the two-dimensional quantum mechanical oscillator, and are orthogonal in the following sense:

$$\int d\mathbf{v}\left[\frac{\exp(-\frac{1}{2}v^2)}{2\pi}\right]E_{nm}^*(\mathbf{v})E_{n'm'}(\mathbf{v}) = \frac{2^{|m|}[(n+|m|)/2]!}{[(n-|m|)/2]!}\delta_{nn'}\delta_{mm'} \tag{2.7}$$

If follows that the velocity distribution function can be expressed in terms of them:

$$f(\mathbf{v},t) = \frac{\exp(-\frac{1}{2}v^2)}{2\pi}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}{}' d_{nm}(t)E_{nm}(\mathbf{v}) \tag{2.8a}$$

with

$$d_{nm}(t) = (-2)^{(n-|m|)/2}\left[\frac{n-|m|}{2}\right]!C_{nm}(t) \tag{2.8b}$$

Alternatively this can be derived applying termwise Fourier inversion to (2.3). For isotropic initial condition it reduces to the known Laguerre series expansion.[1]

From the property $\Phi(-\mathbf{k},t) = \Phi^*(\mathbf{k},t)$ it follows that $d_{n,-m}^*(t) = d_{n,m}(t)$, guaranteeing that (2.8) represents a real-valued function.

Next we derive the equations for the nonisotropic Laguerre moments, that follow from (2.1) and (2.3). For convenience we denote vectors in the plane as complex numbers, i.e., we use the mapping:

$$\mathbf{k} = k(\cos\alpha, \sin\alpha) \mapsto ke^{i\alpha} \tag{2.9a}$$

and consider $\Phi(\mathbf{k},t)$ as a function in the complex plane. It is not difficult to see that the vectorial arguments occuring under the integral sign in (2.1) are mapped as follows:

$$\frac{1}{2}k(\hat{k}+\hat{n}) \mapsto k\cos\left(\frac{\phi-\alpha}{2}\right)\exp\left[i\frac{\phi+\alpha}{2}\right]$$
$$\frac{1}{2}k(\hat{k}-\hat{n}) \mapsto k\cos\left(\frac{\phi+\pi-\alpha}{2}\right)\exp\left[i\frac{\phi+\pi+\alpha}{2}\right] \tag{2.9b}$$

Using this and Eq. (2.3) we can express the first part of the integral (gain term) in (2.1) as follows (introducing $\psi \equiv \phi - \alpha$ as a new integration variable):

$$\begin{aligned}
\text{G.T.} &= \int_0^{2\pi} d\phi\, \bar{\sigma}\left[\cos(\phi-\alpha)\right] \Phi\left[\tfrac{1}{2}k(\hat{k}+\hat{n}), t\right] \Phi\left[\tfrac{1}{2}k(\hat{k}-\hat{n}), t\right] \\
&= \exp\left(-\tfrac{1}{2}k^2\right) \int_0^{2\pi} d\psi\, \bar{\sigma}(\cos\psi) \sum_{n, n', m, m'} (ik)^{n+n'} \left(\cos\frac{\psi}{2}\right)^{n'} \left(\cos\frac{\psi+\pi}{2}\right)^{n} \\
&\quad \times \exp\left[i(m+m')\alpha + i\left(\frac{m+m'}{2}\right)\psi + im\left(\frac{\pi}{2}\right)\right] C_{nm}(t) C_{n'm'}(t)
\end{aligned} \tag{2.10}$$

which can be written in the following form:

$$\text{G.T.} = \exp\left(-\tfrac{1}{2}k^2\right) \sum_{N=0}^{\infty} \sum_{M=-N}^{N}{}' \exp(iM\alpha)(ik)^N G_{NM} \tag{2.11a}$$

with

$$\begin{aligned}
G_{NM} &= \sum_{n=0}^{N} \sum_{m=m_0}^{m_1}{}' \mu_{nm}^{NM} C_{nm}(t) C_{N-n, M-m}(t) \\
m_0 &= \max(-n, M-N+n), \qquad m_1 = \min(n, M+N-n)
\end{aligned} \tag{2.11b}$$

The real-valued coefficients μ_{nm}^{NM} are nonzero only when both $n+m$ and $N+M$ are even, in which case they are given by

$$\mu_{nm}^{NM} = \int_0^{2\pi} d\psi\, \bar{\sigma}(\cos\psi) \left[\cos\frac{\psi}{2}\right]^{N-n} \left[\sin\frac{\psi}{2}\right]^{n} \exp\left[iM\frac{\psi}{2} + i\pi\left[\frac{m}{2} + n\right]\right] \tag{2.11c}$$

By equating coefficients of $k^N e^{iM\alpha}$ after substituting (2.3) and (2.11) into (2.1) we derive the moment equations:

$$\dot{C}_{NM}(t) + \mu_{00}^{00} C_{NM}(t) = \sum_{n=0}^{N} \sum_{m=m_0}^{m_1}{}' \mu_{nm}^{NM} C_{nm}(t) C_{N-m, M-m}(t) \tag{2.12}$$

In case the integral for ψ_{00}^{NM} diverges—as for actual Maxwell molecules—this equation is still valid if we subtract the divergent contributions on both sides (they are equal). This would have followed if we would have taken together the gain term and the loss term from the very beginning.

About the system of moment equations we note the following: for a given model, defined by a collision rate $\sigma(\cos\psi)$ the quantities μ_{nm}^{NM} can be calculated. For a given initial distribution the non-isotropic Laguerre moments at initial time can be found from (2.4). Since the equation with

$\dot{C}_{NM}(t)$ contains $C_{NM}(t)$ linearly and further only moments with lower N values, the system can be solved recursively with respect to N. Every moment consists of a finite (though with N rapidly increasing) number of transients, i.e., terms exponentially decreasing with time.

Numerically the velocity distribution function must be approximated by a finite number of terms in (2.8). In this sense (2.12) together with (2.8) constitutes the solution of the nonlinear Boltzmann equation for Maxwell models.

For isotropic initial conditions the solution reduces to the Laguerre series solution as first discovered by Ernst.[2,1] For that case the equations [in terms of the $d_{nm}(t)$, see (2.8b)] for the only nonzero moments take the form

$$\dot{d}_{2N,0}(t) + \mu_{0,0} d_{2N,0}(t) = \sum_{K=0}^{N} \mu_{N,K} d_{2K,0}(t) d_{2(N-K),0}(t) \tag{2.13a}$$

with

$$\mu_{N,K} = \binom{N}{K} \mu_{2K,0}^{2N,0} = \binom{N}{K} \int_0^{2\pi} \underline{\sigma}(\cos\psi) \left[\sin\frac{\psi}{2}\right]^{2K} \left[\cos\frac{\psi}{2}\right]^{2(N-K)} d\psi \tag{2.13b}$$

The quantities $\mu_{N,K}$ are given by Ernst[1] for a number of models (e.g., for the well-known Tjon–Wu model $\mu_{N,K} = (N+1)^{-1}$). We note that the nonisotropic coefficients μ_{nm}^{NM} can be expressed in terms of the isotropic ones, as follows from (2.11c) and (2.13b):

$$\mu_{nm}^{NM} = (\operatorname{sgn} M)^n \sum_{k=0}^{|M|}{}' \binom{|M|}{k} (-)^{(2n+m+k)/2} \begin{bmatrix} \dfrac{N+|M|}{2} \\ \dfrac{n+k}{2} \end{bmatrix}^{-1} \mu_{(N+|M|)/2,(n+k)/2} \tag{2.14}$$

where the sum is over values of k for which $n+k$ is even and $\operatorname{sgn}(M)$ is -1 when M is negative and $+1$ when M is positive or zero.

The solution to the moment equations (2.12) is given recursively by

$$C_{NM}(t) = \exp(-\lambda_{NM} t) \Bigg[C_{NM}(0) + \int_0^t d\tau \exp(\lambda_{NM}\tau) \times \sum_{n=2}^{N-2} \sum_{m=m_0}^{m_1} \mu_{nm}^{NM} C_{nm}(\tau) C_{N-n,M-m}(\tau) \Bigg] \tag{2.15}$$

where λ_{NM} are the eigenvalues of the linearized Boltzmann equation

$$\lambda_{NM} = \int_0^{2\pi} d\psi\, \bar{\sigma}(\cos\psi) \left\{ 1 - \left[\cos^N\left(\frac{\psi}{2}\right) + \sin^N\left(\frac{\psi}{2}\right) \right] e^{iM\psi/2} \right\} \tag{2.16}$$

The conservation laws (1.6) are translated into

$$\begin{aligned} C_{00}(t) &= 1 \\ C_{1m}(t) &= 0, \qquad m = \pm 1 \end{aligned} \tag{2.17}$$

consistent with (2.12). We note that the normalized value of $C_{00}(t)$ has been assumed in the derivation of (2.12); solving (2.12) with a different value of $C_{00}(0)$, as was done in Ref. 18, is inconsistent with the starting equation (1.1).

To demonstrate the usefulness of the still formal solution (2.15) one has to show that the corresponding series (2.8) converges and that the sum satisfies the Boltzmann equation. In the case of isotropic initial conditions a very elegant theorem to this effect has been given by Bobylev,[17] which has a very simple proof and the advantage that it is valid for all Maxwell models (including Maxwell molecules). For our two-dimensional non-isotropic case a similar theorem can be given. Its proof makes use of the fact that the number

$$C_1 = \max_{\substack{N,M \\ N \geqslant 4}} \frac{1}{\lambda_{NM}} \sum_{n=2}^{N-2} \binom{N/2}{n/2} |\mu_{n0}^{NM}| \tag{2.18}$$

is finite (see Appendix B). The theorem can be formulated as follows: given any set of positive numbers θ_N subject to the condition $\theta_K \theta_{N-K} \leqslant \theta_N$ and a real number $\alpha > 1$, the set of numbers θ_{NM} defined by

$$\theta_{NM} = \frac{\theta_N}{C_1 C_2 (|M| + 1)^\alpha \Gamma(N/2 + 1)} \tag{2.19}$$

with

$$C_2 = 2 \sum_{m=-\infty}^{\infty} (|m| + 1)^{-\alpha}$$

can serve as uniform upperbounds of the moments, i.e., if $|C_{NM}(0)| < \theta_{NM}$ then $|C_{NM}(t)| < \theta_{NM}$. The proof, carried out by induction, is given in Appendix B. There we also show that for $\theta_n = p^n$, $0 < p < 1/2$ the series (2.8a) converges in the mean to a function inside the above mentioned Hilbert space and also that the convergence is uniform in $\mathbf{v}$ and t and the sum is a solution to the Boltmann equation.

3. SOLUTION TO THE THREE-DIMENSIONAL MODEL

For three-dimensional Maxwell models the Fourier-transformed Boltzmann equation has the form

$$\frac{\partial\Phi(\mathbf{k},t)}{\partial t}=\int d\hat{n}\,\bar{\sigma}(\hat{k}\cdot\hat{n})\left\{\Phi\left[\tfrac{1}{2}k(\hat{k}+\hat{n}),t\right]\Phi\left[\tfrac{1}{2}k(\hat{k}-\hat{n}),t\right]-\Phi(\mathbf{k},t)\right\} \tag{3.1}$$

where in polar coordinates the occurring vectors are now given by

$$\begin{aligned}\mathbf{k}&=k(\cos\alpha\sin\beta,\sin\alpha\sin\beta,\cos\beta)\\ \hat{n}&=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)\end{aligned} \tag{3.2}$$

Assuming the existence of the moments of $f(\mathbf{v},t)$ we expand the characteristic function—analogous to what we did in Eq. (2.3)—in terms of an appropriate set of functions:

$$\Phi(\mathbf{k},t)=(4\pi)^{1/2}\exp\left(-\tfrac{1}{2}k^2\right)\sum_{n=0}^{\infty}\sum_{l=0}^{n}{}'\sum_{m=-l}^{l}C_{nlm}(t)(ik)^nY_{lm}(\beta,\alpha) \tag{3.3}$$

In Appendix A the relation between the coefficients $C_{nlm}(t)$ and the velocity distribution function is derived. These coefficients vanish unless $n+l$ is even (we put the prime as a reminder of this fact). The $Y_{lm}(\beta,\alpha)$—also denoted as $Y_{lm}(\hat{k})$—are the well-known spherical harmonic functions.[8]

The $C_{nlm}(t)$ are again generalized moments of the velocity distribution function (see Appendix A):

$$C_{nlm}(t)=\frac{(-)^{(n-l)/2}2^{-(n+l+2)/2}\pi^{1/2}}{\Gamma\left[(n+l+3)/2\right]}\langle E^*_{nlm}(\mathbf{v})\rangle \tag{3.4}$$

where the complete set of basis functions $E_{nlm}(\mathbf{v})$ is defined by

$$\begin{gathered}E_{nlm}(\mathbf{v})=(4\pi)^{1/2}v^l\mathcal{L}^{(l+1/2)}_{(n-l)/2}\left(\tfrac{1}{2}v^2\right)Y_{lm}(\hat{v})\\ n+l\text{ even},\quad 0\leqslant n<\infty,\quad 0\leqslant l\leqslant n,\quad -l\leqslant m\leqslant l\end{gathered} \tag{3.5}$$

These functions (Bunnett functions[15]) are eigenfunctions of the three-dimensional quantum mechanical oscillator problem (treated in spherical coordinates). They have the orthogonality property

$$\int d\mathbf{v}\left[\frac{\exp\left(-\tfrac{1}{2}v^2\right)}{(2\pi)^{3/2}}\right]E^*_{nlm}(\mathbf{v})E_{n'l'm'}(\mathbf{v})=\frac{2^{l+1}\pi^{-1/2}\Gamma\left[(n+l+3)/2\right]}{\left[(n-l)/2\right]!}\delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{3.6}$$

From this and (3.4) it follows that the velocity distribution function can be expressed in terms of them [alternatively this can be derived by Fourier inverting (3.3) term by term]

$$f(\mathbf{v},t)=\frac{\exp\left(-\frac{1}{2}v^2\right)}{(2\pi)^{3/2}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}{}'\sum_{m=-l}^{l}d_{nlm}(t)E_{nlm}(\mathbf{v}) \tag{3.7}$$

with

$$d_{nlm}(t)=\left(\frac{n-l}{2}\right)!\,(-2)^{(n-l)/2}C_{nlm}(t) \tag{3.8}$$

Now we derive the set of equations for these (nonisotropic Laguerre) moments. Using (3.3) we express the gain term (G.T.) in terms of the $C_{nlm}(t)$. To this end we define angles $\theta_{\pm}$ and $\phi_{\pm}$ such that

$$\tfrac{1}{2}k(\hat{k}\pm\hat{n})=k\left[\tfrac{1}{2}(1\pm\hat{k}.\hat{n})\right]^{1/2}(\cos\phi_{\pm}\sin\theta_{\pm},\sin\phi_{\pm}\sin\theta_{\pm},\cos\theta_{\pm}) \tag{3.9}$$

Now the gain term may be written as

$$\begin{aligned}\text{G.T.}=(4\pi)\exp\left(-\frac{1}{2}k^2\right)\sum_{\substack{nlm\\n'l'm'}}\int d\hat{n}\,\bar{\sigma}(\hat{k}\cdot\hat{n})\left[\frac{1}{2}(1+\hat{n}\cdot\hat{k})\right]^{n/2}\\ \times\left[\frac{1}{2}(1-\hat{n}\cdot\hat{k})\right]^{n'/2}(ik)^{n+n'}Y_{lm}(\theta_+,\phi_+)Y_{l'm'}(\theta_-,\phi_-)C_{nlm}(t)C_{n'l'm'}(t).\end{aligned} \tag{3.10}$$

We choose a new, rotated coordinate frame in $\hat{n}$ space, such that the z' axis is directed along $\hat{k}$. In the plane orthogonal to $\hat{k}$ the x' axis may be chosen arbitrarily. Let $\theta'_{\pm}$ and $\phi'_{\pm}$ be the polar angles corresponding to $\hat{k}\pm\hat{n}$ in the new frame. The Y_{lm} in the old frame can be expressed in terms of the new ones as[8]

$$Y_{lm}(\theta_{\pm},\phi_{\pm})=\sum_{m'=-l}^{l}D_{mm'}^{(l)*}(\alpha,\beta)Y_{lm'}(\theta'_{\pm},\phi'_{\pm}) \tag{3.11}$$

The quantities $D_{mm'}^{(l)}$ are matrix elements corresponding to an l-dimensional representation of the rotation that maps the z axis onto $\hat{k}$. We denote the polar angles in the new frame corresponding to $\hat{n}$ by θ' and ϕ'. The Jacobian of the transformation of variables is of course equal to unity. Since the z' axis is along $\hat{k}$ the following relations are valid:

$$\begin{aligned}&\hat{k}\cdot\hat{n}=\cos\theta'\\ &\theta'_+=\tfrac{1}{2}\theta',\qquad \theta'_-=\tfrac{1}{2}(\pi-\theta')\\ &\phi'_+=\theta'\qquad \phi'_-=\phi'+\pi\end{aligned} \tag{3.12}$$

In the following we drop all primes (the new coordinate frame is understood throughout). The gain term can now be written as

$$\mathrm{G.T.} = (4\pi)\exp\left(-\frac{1}{2}k^2\right) \sum_{\substack{nlmp\\ n'l'm'p'}} (ik)^{n+n'} \int d\phi\, d\cos\theta$$
$$\times \left[\bar{\sigma}(\cos\theta)\left(\cos\frac{\theta}{2}\right)^{n}\left(\sin\frac{\theta}{2}\right)^{n'} Y_{lp}\left(\frac{\theta}{2},\phi\right) Y_{l'p'}\left(\frac{\pi-\theta}{2},\phi+\pi\right)\right]$$
$$\times D_{mp}^{(l)*}(\alpha,\beta) D_{m'p'}^{(l')*}(\alpha,\beta) C_{nlm}(t) C_{n'l'm'}(t) \tag{3.13}$$

The only $\hat{k}$ dependence in this formula is through the matrix elements $D_{mm'}^{(l)}(\alpha,\beta)$, outside the integral sign. The ϕ integration can be done straightforwardly, the ϕ dependence of $Y_{lp}(\theta,\phi)$ being through a simple factor $e^{ip\phi}$:

$$\int_0^{2\pi} d\phi\, Y_{lp}\left(\frac{\theta}{2},\phi\right) Y_{l'p'}\left(\frac{\pi-\theta}{2},\phi+\pi\right)$$
$$= (2\pi)\delta_{p+p',0}(-)^{p'} Y_{lp}\left(\frac{\theta}{2},0\right) Y_{l'p'}\left(\frac{\pi-\theta}{2},0\right)$$
$$= (2\pi)\delta_{p+p',0} Y_{l|p|}\left(\frac{\theta}{2},0\right) Y_{l'|p'|}\left(\frac{\pi-\theta}{2},0\right) \tag{3.14}$$

where we used $Y_{lm}(\theta,0)=(-)^m Y_{l,-m}(\theta,0)$ in the second equality. Because of the Kronecker delta in (3.14) we must have $p=-p'$ in (3.13). Then we can use the well-known Clebsch–Gordan series[8] to rewrite the product of matrix elements:

$$D_{mp}^{(l)}(\alpha,\beta) D_{m',-p}^{(l')}(\alpha,\beta)$$
$$= \sum_{L=|l-l'|}^{l+l'} \langle ll'mm' \mid ll'L, m+m'\rangle \langle ll'p,-p \mid ll'L0\rangle D_{m+m',0}^{(L)}(\alpha,\beta)$$
$$= \sum_{L=|l-l'|}^{l+l'} \langle ll'mm' \mid ll'L, m+m'\rangle \langle ll'p,-p \mid ll'L0\rangle \left(\frac{4\pi}{2L+1}\right)^{1/2}$$
$$\times Y_{L,m+m'}^{*}(\beta,\alpha) \tag{3.15}$$

The definition of the Clebsch–Gordan coefficients used here can be found in Ref. 8. Explicit expressions and a scheme for evaluating them numerically fast are given by Rose.[8] Now the gain term takes the form

$$\mathrm{G.T.} = (4\pi)^{1/2}\exp\left(-\tfrac{1}{2}k^2\right) \sum_{N=0}^{\infty} \sum_{L=0}^{N}{}' \sum_{M=-L}^{L} G_{NLM}(ik)^N Y_{LM}(\beta,\alpha) \tag{3.16a}$$

with

$$G_{NLM} = \frac{8\pi^2}{(2L+1)^{1/2}} \sum_{n=0}^{N} \sum_{l=0}^{n}{}' \sum_{l'=0}^{N-n} \sum_{m=m_0}^{m_1} \sum_{p=-p_0}^{p_0} \langle ll'm, M-m \,|\, ll'LM\rangle$$
$$\times \langle ll'p, -p|ll'L0\rangle$$
$$\times \int_0^{\pi} d\cos\theta \Big[\bar{\sigma}(\cos\theta)\Big(\cos\frac{\theta}{2}\Big)^n \Big(\sin\frac{\theta}{2}\Big)^{N-n}$$
$$\times Y_{l|p|}\Big(\frac{\theta}{2}, 0\Big) Y_{l'|p|}\Big(\frac{\pi-\theta}{2}, 0\Big)\Big] C_{nlm}(t) C_{N-n,l',M-m}(t) \tag{3.16b}$$

with $m_0 = \max(-l, M-l')$, $m_1 = \min(l, M+l')$ and $p_0 = \min(l, l')$.[3] The moment equations can now be written as

$$\dot{C}_{NLM}(t) + \mu_{0000}^{000} C_{NLM}(t) = \sum_{n=0}^{N} \sum_{l=0}^{n}{}' \sum_{l'=0}^{N-n}{}' \sum_{m=m_0}^{m_1} \mu_{nll'm}^{NLM} C_{nlm}(t) C_{N-n,l',M-m}(t) \tag{3.17a}$$

We remind the reader that there are only terms for which both $l+n$ and $l'+N-n$ are even (this is indicated by the two primes). There is however, another restriction on l and l', namely, $|l-l'| \leqslant L \leqslant l+l'$ as follows from (3.15). Outside this region in the (l, l') plane we define $\mu_{nll'm}^{NLM} = 0$.

Like in the two-dimensional case, if μ_{00L0}^{NLM} and μ_{NL0m}^{NLM} diverge (as is the case for Maxwell molecules) their contributions have to be subtracted on both sides and a finite part remains. This would have been the result if we would have taken together gain and loss terms from the very beginning.

The coefficients $\mu_{nll'm}^{NLM}$ are given by

$$\mu_{nll'm}^{NLM} = \frac{8\pi^2}{(2L+1)^{1/2}} \langle ll'm, M-m|ll'LM\rangle \sum_{p=-\min(l,l')}^{\min(l,l')} \langle ll'p, -p|ll'L0\rangle$$
$$\times \int_0^{\pi} d\cos\theta\, \bar{\sigma}(\cos\theta)\Big(\cos\frac{\theta}{2}\Big)^n \Big(\sin\frac{\theta}{2}\Big)^{N-n}$$
$$\times Y_{l|p|}\Big(\frac{\theta}{2}, 0\Big) Y_{l'|p|}\Big(\frac{\pi-\theta}{2}, 0\Big) \tag{3.17b}$$

The appearance of the Clebsch–Gordan coefficient outside the summation

[3] For $N+L$ odd, G_{NLM} vanishes. This follows from the fact that both $n+l$ and $N-n+l'$ have to be even and the symmetry property:

$$\langle ll'p, -p \,|\, ll'L0\rangle = (-)^{l+l'+L} \langle ll' -p, p \,|\, ll'L0\rangle \qquad \text{(see Ref. 8)}$$

is related to the invariance of the nonlinear Boltzmann equation under rotations.

Using the explicit form of $Y_{lp}(\theta/2, 0)$ in terms of $\sin(\theta/2)$ and $\cos(\theta/2)$ it is possible to express the coefficients $\mu_{nll'm}^{NLM}$ in terms of the isotropic ones (i.e., the ones having $L = M = l = l' = m = 0$). We find

$$\mu_{nll'm}^{NLM} = \frac{2\pi}{(2L+1)^{1/2}} \langle ll'm, M-m | ll'LM \rangle 2^{-(l+l')} \sum_{p=-\min(l,l')}^{\min(l,l')}$$

$$\times \langle ll'p, -p \,|\, ll'L0 \rangle \left[(2l+1)(2l'+1) \frac{(l-|p|)!\,(l'-|p|)!}{(l+|p|)!\,(l'+|p|)!} \right]^{1/2}$$

$$\times \sum_{k=|p|}^{l}{}' \sum_{k'=|p|}^{l'}{}' \frac{(-)^{(l-k+l'-k')/2}(l+k)!\,(l'+k')!}{[(k+l)/2]!\,[(k'+l')/2]!\,[(l-k)/2]!}$$

$$\times \frac{1}{[(l'-k')/2]!\,(k-|p|)!\,(k'-|p|)!}$$

$$\times \begin{pmatrix} \frac{N+k+k'}{2} \\ \frac{n+k}{2} \end{pmatrix}^{-1} \mu_{(N+k+k')/2,(n+k)/2} \tag{3.18a}$$

The sums over k and k' contain only terms for which $k + l$ and $k' + l'$ are even. The coefficients $\mu_{n,k}$ defined as

$$\mu_{n,k} = (2\pi)\binom{n}{k}\int_0^{\pi} d(\cos\theta)\,\bar{\sigma}(\cos\theta)\left(\cos\frac{\theta}{2}\right)^{2k}\left(\sin\frac{\theta}{2}\right)^{2(n-k)}$$

$$= \binom{n}{k}\mu_{2k,000}^{2n,00} \tag{3.18b}$$

have been calculated for a number of models[1] in explicit form.

As in the two-dimensional case the moment equations are recursively soluble for given initial conditions $C_{NLM}(0)$, that have to be determined from (3.4). At a later time the velocity distribution function is then given by (3.7).

We note that the eigenfunctions and eigenvalues of the corresponding equation, which are long known, can be derived from the above results in a simple way. In the three-dimensional case we simply substitute $C_{NLM}(t) = C_{NLM}(\infty) + b_{NLM}(t)$ with $C_{NLM}(\infty) = \delta_{N0}\delta_{L0}\delta_{M0}$ into the moment equations (3.17a) and neglect terms quadratic in the $b_{NLM}(t)$. In that approximation the $b_{NLM}(t)$ decay exponentially [$\sim\exp(-\lambda_{NL}t)$] and the eigenvalue occurring in the exponent is given by

$$\begin{aligned}\lambda_{NL} &= \mu_{0000}^{000}(1+\delta_{N,0}) - \left(\mu_{00L0}^{NLM} + \mu_{NL0M}^{NLM}\right)\\ &= (2\pi)\int_0^{\pi} d\cos\theta\,\bar{\sigma}(\cos\theta)\left\{1+\delta_{N,0} - \left(\sin\frac{\theta}{2}\right)^N P_L\left(\sin\frac{\theta}{2}\right)\right.\\ &\qquad \left. - \left(\cos\frac{\theta}{2}\right)^N P_L\left(\cos\frac{\theta}{2}\right)\right\} \end{aligned} \tag{3.19}$$

a well-known result. The eigenfunctions are given by (3.5).

We note that the five lowest eigenvalues λ_{00}, λ_{1m} and λ_{20} are all zero. As a consequence of this the quantities $C_{000}(t)$, $C_{11M}(t)$ and $C_{200}(t)$ are conserved, as follows from (3.17). [Note that if we would not had implicitly normalized $C_{000}(t)$ to unity already in (3.1), the term with $C_{NLM}(t)$ on the left-hand side would have been multiplied by $C_{000}(t)$.]

The nonisotropic eigenvalues (with $L \neq 0$) can be expressed in terms of the isotropic ones (with $L = 0$) as

$$\lambda_{NL} = \sum_{k=0}^{L} a_{k,L}\lambda_{N+k,0} \tag{3.20}$$

where $a_{k,L}$ is the coefficient of x^k in $P_L(x)$.

Table I. Eigenvalues in the Three-Dimensional Isotropic Scattering Model

N/L	0	1	2	3	4	5	6	7	8	9	10
0	0										
1	*	0									
2	0	*	$\frac{1}{2}$								
3	*	$\frac{1}{3}$	*	$\frac{3}{4}$							
4	$\frac{1}{3}$	*	$\frac{7}{12}$	*	$\frac{7}{8}$						
5	*	$\frac{1}{2}$	*	$\frac{3}{4}$	*	$\frac{15}{16}$					
6	$\frac{1}{2}$	*	$\frac{13}{20}$	*	$\frac{41}{48}$	*	$\frac{31}{32}$				
7	*	$\frac{3}{5}$	*	$\frac{23}{30}$	*	$\frac{11}{12}$	*	$\frac{63}{64}$			
8	$\frac{3}{5}$	*	$\frac{7}{10}$	*	$\frac{17}{20}$	*	$\frac{61}{64}$	*	$\frac{127}{128}$		
9	*	$\frac{2}{3}$	*	$\frac{11}{14}$	*	$\frac{29}{32}$	*	$\frac{187}{192}$	*	$\frac{255}{256}$	
10	$\frac{2}{3}$	*	$\frac{31}{42}$	*	$\frac{191}{224}$	*	$\frac{181}{192}$	*	$\frac{757}{768}$	*	$\frac{511}{512}$

This follows directly from (3.19). For the (Krook–Wu) isotropic scattering model [where $\lambda_{N,0} = (N-2)/(N+2)$] we used this to calculate the first few eigenvalues (see Table I). Note that λ_{NL} increases with increasing L and fixed N and also with increasing N and fixed L.

Like in the two-dimensional case it can be proved that for a class of initial conditions the series (3.7) converges both uniformly in $\mathbf{v}$ and t and in the mean (see Appendix C) to a solution of the Boltzmann equation.

Now the answer can be given to the interesting equation: which is the lowest linear mode decaying faster than (or as fast as) the slowest term due to nonlinearity in the solution of the full equation. In the isotropic scattering model this is the mode with $N = 9$, $L = 1$. The linear $N = 9$ mode decays as $\exp(-\lambda_{91}t) = \exp(-\frac{2}{3}t)$, whereas the equation for c_{800} contains the (nonlinear) term $\mu_{0000}^{400}c_{400}^2$, which induces a term in its solution proportional to $\exp(-2\lambda_{40}t) = \exp(-\frac{2}{3}t)$. This is the slowest nonlinear contribution to any moment. For more general cross sections one finds that when using the solution of the linear equation to describe the approach to equilibrium one should leave out all the modes with eigenvalues higher than twice the lowest, since beyond this one the linear modes decay faster than the slowest term due to nonlinearity, in the solution to the full equation. This has been noted before by Kac.[14]

4. ORDINARY MOMENTS AND POLYNOMIAL MOMENTS

We have discussed the nonlinear Boltzmann equation for Maxwell models in terms of the nonisotropic Laguerre moments, using spherical coordinates. These moments proved to be very useful if the aim is retrieving the velocity distribution function at a later time given its value at initial time (in the spatially homogeneous case). The moment equations turned out to be recursively soluble and the velocity distribution function could be directly expressed in terms of these moments and the corresponding basis functions.

In the spatially inhomogeneous case the procedure does not go through any longer, since the moment equations couple in the wrong direction. For that case the ordinary moments are more important, since they yield the hydrodynamic equations. These equations have been derived long before.[4] Closely related to these ordinary moments are certain (Hermite) polynomial moments, being linear combinations of the former and satisfying the same equations (as we shall show below). Knowledge of them would also permit retrieval of the distribution function.

The equations for the ordinary moments can be derived within the Fourier transform method. Since the equations themselves are already known we merely sketch the method. The ordinary Taylor expansion of $e^{i\mathbf{k}\cdot\mathbf{v}}$ induces an expansion of the characteristic function:

$$\Phi(\mathbf{k},t)=\langle e^{i\mathbf{k}\cdot\mathbf{v}}\rangle=\sum_{n_1n_2n_3}\langle v_1^{n_1}v_2^{n_2}v_3^{n_3}\rangle\frac{(ik_1)^{n_1}(ik_2)^{n_2}(ik_3)^{n_3}}{n_1!\,n_2!\,n_3!}$$
$$\mathbf{v}=(v_1,v_2,v_3),\qquad \mathbf{k}=(k_1,k_2,k_3) \tag{4.1}$$

The coefficients $\langle v_1^{n_1}v_2^{n_2}v_3^{n_3}\rangle\equiv M_{n_1n_2n_3}(t)$ are ordinary moments of the velocity distribution function. What one has to do is substitute this expression into (3.1) and work out the integral over $\hat{n}$. The calculations are laborious—at least in three dimensions—but the procedure turns out to be possible. For its result we refer to Ref. 4 (note that in this reference tensor moments are used, with a slighly different definition).

The characteristic function can be expanded in still a different manner in terms of polynomial moments, using the fact that the generating function of Hermite polynomials is the exponential function:

$$\Phi(\mathbf{k},\mathbf{t})=\langle e^{i\mathbf{k}\cdot\mathbf{v}}\rangle=e^{-(1/2)k^2}\sum_{m_1m_2m_3}\langle He_{m_1}(v_1)He_{m_2}(v_2)He_{m_3}(v_3)\rangle\times\frac{(ik_1)^{n_1}(ik_2)^{n_2}(ik_3)^{n_3}}{n_1!\,n_2!\,n_3!} \tag{4.2}$$

The polynomials $He_n(x)$ are defined in Ref. 9. The coefficients $\langle He_{m_1}(v_1)He_{m_2}(v_2)He_{m_3}(v_3)\rangle\equiv H_{m_1m_2m_3}(t)$ are the so-called Hermite moments. The equations for the first few closely related tensor Hermite moments have been derived by Grad.[3]

Here we note a relation between the ordinary moments and the Hermite moments. It follows from (3.1) that for every solution $\Phi(\mathbf{k},t)$ the function $\exp(\frac{1}{2}k^2)\Phi(\mathbf{k},t)$ is also a solution (though not satisfying the same boundary conditions). Comparing (4.2) and (4.1) we see that the Taylor series of $\exp(\frac{1}{2}k^2)\Phi(\mathbf{k},t)$ follows from that of $\Phi(\mathbf{k},t)$ itself if we replace the ordinary moments $M_{n_1n_2n_3}(t)$ by the Hermite moments $H_{n_1n_2n_3}(t)$. Therefore both systems of moments satisfy exactly the same set of equations (but with different boundary conditions). This relation was noted before by Ernst[1,2] in the isotropic case.

The equations for the above moments are also recursively soluble in the spatially homogeneous case. However, this is more difficult than before with the nonisotropic Laguerre moments, since now the equation with $\dot{M}_{n_1n_2n_3}(t)$ contains all moments $M_{n_1'n_2'n_3'}(t)$ with $n_1'+n_2'+n_3'\leqslant n_1+n_2+n_3$. This means that at every step an extra linear transformation has to be performed.

5. DISCUSSION

In this paper we have mainly discussed the nonlinear Boltzmann equation for Maxwell models using spherical coordinates and the corresponding (generalized) moments. It turns out that in the spatially homogeneous case these moments are the suitable ones if one wants to construct the velocity distribution function at all times, given it at initial time.

The moments can be found from a recursively soluble set of equations and the distribution function can be expressed in terms of them. This result is an extension of that for isotropic initial conditions, first derived by Ernst.[2] It is valid within the Hilbert space of functions for which $\int |f|^2 e^{(1/2)v^2} d\mathbf{v}$ is finite.[4]

The moment equations derived here are as explicit as possible. All the coefficients have been determined in terms of integrals over the cross section. Like in Ref. 2 the method of the present paper makes use of Fourier transformation. This makes the calculations much easier than with any of the older methods, where all manipulations had to be done on the collision integral in the velocity representation. The method also allows one to construct the equations for the ordinary moments—relevant to the hydrodynamics—and those for the (Hermite) polynomial moments. The former have been derived by Ikenberry and Truesdell[4] in explicit form, the latter have been considered by Grad,[3] who explicitly constructed the first few of them. Unlike here, these authors use tensorial moments with multi-indices, the number of indices increasing with the order of the moments. Here we note that the two types of moments, as defined here, satisfy an identical set of equations, though with different initial conditions of course. This was shown before for isoptropic initial conditions.[2]

Closest to our three-dimensional result is that of Kumar[15] (see Introduction). This author compares the use of the different basis functions and he arrives at the conclusion that the Burnett functions (3.5) are the most economical ones.

The practical importance of the result is somewhat limited. Firstly it applies to Maxwell models only and it seems impossible to extend it to other types of models (as an exact solution). Secondly, in actual (numerical) calculations the solution can only be used at not too high values of the energy, where not too many terms in the series are needed.[5] However, in the isotropic case features like the overshoot phenomenon (discovered by

[4] A modification of the result (see Ref. 10) yields the general solution in the (greater) space of (positive) functions for which $\lim_{v\to\infty} f(\mathbf{v})v^k = 0$ (for every k). Special solutions even outside this larger space have been found too.[6,7]

[5] The numerical convergence of the series may be improved by using generalized Padé resummation,[13] the modified Laguerre series,[1] or both.

Tjon,[11] who numerically integrated the nonlinear Boltzmann equation) could be reproduced using the series solution.[1,12]

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APPENDIX A. DERIVATION OF THE EXPANSIONS OF THE CHARACTERISTIC FUNCTION

First we quote some necessary standard formulas of mathematical analysis[8,9]:

$$\exp(ix\cos y) = \sum_{m=-\infty}^{\infty} i^{|m|} J_{|m|}(x) \exp(imy) \tag{A1}$$

$$\exp(i\mathbf{k}\cdot\mathbf{v}) = (4\pi)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kr) Y_{lm}(\hat{k}) Y_{lm}^*(\hat{v}) \tag{A2}$$

$$j_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{l+1/2}(z) \tag{A3}$$

$$J_{|m|}\left[2(xz)^{1/2}\right] = (xz)^{(1/2)|m|}\exp(-z)\sum_{n=0}^{\infty}\frac{\mathcal{L}_n^{(|m|)}(x) z^n}{\Gamma(n+|m|+1)} \tag{A4}$$

In here, $J_{|m|}(x)$ is the ordinary Bessel function and $j_l(z)$ the spherical Bessel function. In (A2), $\mathbf{k}$ and $\mathbf{v}$ are three-dimensional vectors. The $\mathcal{L}_n^{(m)}(x)$ are generalized Laguerre polynomials.

The expansion of the characteristic function is derived as follows: in the two-dimensional case we use (A1) together with (A4) to expand $\exp(i\mathbf{k}\cdot\mathbf{v})$:

$$\begin{aligned}\exp(i\mathbf{k}\cdot\mathbf{v}) &= \exp\left[ikv\cos(\phi-\alpha)\right] \\ &= \exp\left(-\frac{1}{2}k^2\right)\sum_{n=0}^{\infty}\sum_{m=-n}^{n}{}' \frac{i^{|m|}\exp\left[im(\phi-\alpha)\right]k^n\left(\frac{1}{2}\right)^{(n+|m|)/2}}{(n+|m|/2)!} \\ &\quad \times \mathcal{L}_{(n-|m|)/2}^{(|m|)}\left(\frac{1}{2}v^2\right)v^{|m|}\end{aligned} \tag{A5}$$

Averaging this with respect to the velocity distribution function yields (2.3) and (2.4). In the derivation it becomes clear that only powers k^n occur for which $n+m$ is even. This is indicated by the prime.

In the three-dimensional case we combine (A2), (A3), and (A4) into a single formula:

$$\exp(i\mathbf{k}\cdot\mathbf{v}) = \pi^{3/2}\exp\left(-\frac{1}{2}k^2\right)\sum_{n=0}^{\infty}\sum_{l=0}^{n}{}' \sum_{m=-l}^{l} \frac{(ik)^n(-)^{(n-l)/2}2^{(2-l-n)/2}}{\Gamma[(n+l+3)/2]}$$

$$\times Y_{lm}(\hat{k})Y^*_{lm}(\hat{v})\mathcal{L}^{(l+1/2)}_{(n-l)/2}\left(\frac{1}{2}v^2\right)v^l \tag{A6}$$

The prime now indicates that $l+n$ has to be even. Averaging with respect to the three-dimensional velocity distribution function yields (3.3) and (3.4).

APPENDIX B. CONVERGENCE OF THE SERIES (TWO-DIMENSIONAL CASE)

B.1. Derivation of an Auxiliary Inequality

We first prove the crucial property $C_1 < \infty$, where

$$C_1 = \max_{\substack{N,M\\N\geqslant 4}} \frac{1}{\lambda_{NM}} \sum_{n=2}^{N-2} \begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix} |\mu^{NM}_{no}| \tag{B1}$$

and then proceed to the actual proof of the theorem of Section 2. We use the abbreviations: $\bar{\sigma} \equiv \bar{\sigma}(\cos\psi)$, $s \equiv \sin(\psi/2)$, $c \equiv \cos(\psi/2)$, and

$$f^N_n = \begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix} (c^n s^{N-n} + s^n c^{N-n})$$

Note from (2.11c) and (2.16)

$$\frac{1}{\lambda_{NM}} \sum_{n=2}^{N-2} \begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix} |\mu^{NM}_{n0}| \leqslant \frac{2}{\lambda_N}\int_0^{\pi}\bar{\sigma}\,d\psi \sum_{n=2}^{N-2} f^N_n \tag{B2}$$

with $\lambda_N \equiv 2\int_0^{\pi}\bar{\sigma}(1-c^N-s^N)\,d\psi$. We have to show that the right-hand side of this remains finite in the limit $N\to\infty$. For odd values of n we have $f^N_n \leqslant f^N_{n+1} + f^N_{n-1}$ except at a local maximum of f^N_n as a function of n (there are two such maxima). Hence we have

$$\sum_{n=2}^{N-2} f^N_n \leqslant 3\sum_{\substack{n=2\\n\text{ even}}}^{N-2} f^N_n + 2\max_n f^N_n \tag{B3}$$

Using, e.g., the method of steepest descent one shows that the last term does not contribute to the sum in the limit $N \to \infty$ (i.e., $\lim_{N\to\infty} \int d\psi\, \bar{\sigma}$ $[\max_{n<N} f_n^N] = 0$).

We distinguish between even and odd values of N. For even N we use the binomial theorem:

$$3 \sum_{\substack{n=2 \\ (n \text{ even}) \\ (N \text{ even})}}^{N-2} f_n^N = 3(1 - s^N - c^N). \tag{B4}$$

For odd N we use the fact that

$$\begin{pmatrix} \frac{N}{2} \\ \frac{n}{2} \end{pmatrix} \leqslant A \begin{pmatrix} \frac{N-1}{2} \\ \frac{n}{2} \end{pmatrix}$$

for $2 \leqslant n \leqslant (N-1)/2$ (where A is a constant) and the binomial theorem again:

$$3 \sum_{\substack{n=2 \\ (n \text{ even}) \\ (N \text{ odd})}}^{N-2} f_n^N = 6 \sum_{\substack{n=2 \\ (n \text{ even}) \\ (N \text{ odd})}}^{(1/2)(N-1)} f_n^N \leqslant 6A \sum_{\substack{n=2 \\ (n \text{ even}) \\ (N \text{ odd})}}^{(1/2)(N-1)} \begin{pmatrix} \frac{N-1}{2} \\ \frac{n}{2} \end{pmatrix} (c^n s^{N-n} + s^n c^{N-n})$$

$$< 6A(s+c)(1 - c^{N-1} - s^{N-1}) < 12A(1 - c^N - s^N) \tag{B5}$$

So for both even and odd N values we have

$$\frac{2}{\lambda_N} \int_0^{\pi} \bar{\sigma}\, d\psi \sum_{n=2}^{N-2} f_n^N \leqslant \frac{\text{const}}{\lambda_N} \int_0^{\pi} (1 - c^N - s^N) \bar{\sigma}\, d\psi$$

$$= \text{const} \tag{B6}$$

This proves the desired property $C_1 < \infty$, needed in the proof of the next section.

B.2. Proof of the Theorem of Section 2

$$|C_{NM}(0)| < \theta_{NM} \Rightarrow |C_{NM}(t)| < \theta_{NM} \tag{B7}$$

We choose $C_{NM}(0)$ satisfying the requirement $|C_{NM}(0)| < \theta_{NM}$. For $N \leqslant 3$ the above property trivially holds, since the lower moments decay with a single exponential. If we assume it to hold for all n, m with $n < N$ we can show that it also holds for $n = N$ by the following string of inequalities:

$$\begin{aligned}|C_{NM}(t)| &\leqslant \theta_{NM}\exp(-\lambda_{NM}t) + \frac{1-\exp(-\lambda_{NM}t)}{\lambda_{NM}} \sum_{n=2}^{N-2} |\mu_{n0}^{NM}|\theta_{nm}\theta_{N-n,M-m} \\ &\leqslant \exp(-\lambda_{NM}t)\theta_{NM} + \left[1-\exp(-\lambda_{NM}t)\right]\theta_{NM} \\ &\quad\times\left\{\frac{1}{C_1\lambda_{NM}} \sum_{n=2}^{N-2} |\mu_{n0}^{NM}| \begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix}\right\} \\ &\quad\times\left\{\frac{1}{C_2}\sum_{m=-\infty}^{\infty}\left[\frac{|M|+1}{(|m|+1)(|M-m|+1)}\right]^{\alpha}\right\} \\ &\leqslant \theta_{NM} \end{aligned} \tag{B8}$$

where we used the induction assumption [applied to (2.15)] in the first inequality and the form of the θ_{nm} in the second one (together with the property $\theta_k\theta_{n-k} \leqslant \theta_n$). The expressions between curly brackets are both bounded by unity [see (2.18) and (2.19)]. Hence by induction our theorem is valid.

B.3. Convergence of the Series Expansion

B.3.1. Convergence in the Mean

For convergence in the mean of the series (2.8) it is required that the norm (squared):

$$\|f\|^2 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 2^n \left(\frac{n+|m|}{2}\right)! \left(\frac{n-|m|}{2}\right)! \, |C_{nm}(t)|^2 \tag{B9}$$

remains finite for all times. If we replace $|C_{nm}(t)|$ by its upperbound θ_{nm} and choose $\theta_n = p^n$, $0 < p < \frac{1}{2}$ in (2.19) it is not difficult to show this, using:

$$\frac{\left[(n+|m|)/2\right]!\left[(n-|m|)/2\right]!}{\Gamma(n/2+1)^2} \leqslant \begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} \leqslant 2^n \tag{B10}$$

B.3.2. Uniform Convergence

To prove uniform convergence of the series (2.8) in $\mathbf{v}$ and t we have to show that the tail of this summation

$$\begin{aligned} T_N(v,t) = \frac{e^{-(1/2)v^2}}{2\pi} \sum_{n=N}^{\infty} {\sum_{m=-n}^{n}}' (-2)^{(n-|m|)/2} \\ \times \left(\frac{n-|m|}{2}\right)! \, C_{nm}(t) v^{|m|} \mathcal{L}_{(n-|m|)/2}^{(|m|)}\left(\frac{1}{2}v^2\right) e^{im\beta} \end{aligned} \tag{B11}$$

is uniformly bounded as $N \to \infty$, i.e., $\lim_{N\to\infty}\{\sup_{\mathbf{v}}|T_N(\mathbf{v},t)|\} = 0$. To this end we replace $C_{nm}(t)$ by their bounds θ_{nm}, and the same is done with the Laguerre polynomials, using[9]

$$|\mathfrak{L}^{(l)}_{(n-l)/2}(x)| \leqslant \frac{[(n+l)/2]!}{[(n-l)/2]!\,l!}\, e^{x/2} \tag{B12}$$

With $x = (1/2)v^2$ we find

$$|T_N(\mathbf{v},t)| \leqslant \frac{e^{-x/2}}{\pi} \sum_{n=N}^{\infty} \frac{2^{n/2}\theta_n}{\Gamma[(n/2)+1]} \sum_{m=0}^{n}{}' \frac{[(m+n)/2]!}{m!} x^{m/2} \tag{B13}$$

We extend the m summation to infinity and write $[(m+n)/2]!$ as an integral:

$$\begin{aligned} &\sum_{m=0}^{\infty}{}' \frac{[(m+n)/2]!}{m!} x^{m/2} e^{-x/2} \\ &\quad = \sum_{m=0}^{\infty}{}' \frac{1}{m!} \int_0^\infty dy\, e^{-y} y^{(m+n)/2} x^{m/2} e^{-x/2} \\ &\quad < \int_0^\infty dy\, e^{-y} y^{n/2} e^{(xy)^{1/2} - x/2} < \int_0^\infty dy\, e^{-y/2} y^{n/2} \\ &\quad = 2^{n/2+1}\Gamma\left(\frac{n}{2}+1\right) \end{aligned} \tag{B14}$$

We have eliminated the x dependence by replacing the integrand by its maximum with respect to x. Thus we have found

$$|T_N(\mathbf{v},t)| \leqslant \frac{2}{\pi} \sum_{n=N}^{\infty} 2^n\theta_n = C_N \tag{B15}$$

For $\theta_n = p^n$ with $0 < p < 1/2$ we have $\lim_{N\to\infty} C_N = 0$ and the original series converges uniformly in $\mathbf{v}$ and t. A slight modification of the derivation yields that if we choose $\theta_n = (\frac{1}{2} - \delta)^n$, $0 < \delta < \frac{1}{2}$ a positive number ϵ can be found such that $|T_N(\mathbf{v},t)|$ is bounded by $C_N e^{-\epsilon x}$. This is enough to justify all interchanges of limits needed to show that the sum of the series indeed solves the Boltzmann equation.

APPENDIX C: CONVERGENCE OF THE SERIES (THREE-DIMENSIONAL CASE)

The proof of the convergence of the series solution (3.7) for all times—for a class of initial distributions—is analogous to that for the two-dimensional case. We give only an outline of it. Again a crucial property is needed:

$$C_1 = \max_{\substack{NLM \\ ll'm}} \frac{1}{\lambda_{NL}} \sum_{n=2}^{N-2} \begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix} \frac{2L+1}{(2l+1)(2l'+1)} |\mu_{nll'm}^{NLM}| < \infty \tag{C1}$$

with $\mu_{nll'm}^{NLM}$ as in (3.17b) and λ_{NL} in (3.19). The proof of this makes use of the orthonormality property of the Clebsch–Gordan coefficients:

$$\sum_m |\langle ll'm\, M-m \,|\, ll'LM \rangle|^2 = 1 \tag{C2}$$

implying

$$|\langle ll'm\, M-m \,|\, ll'LM \rangle| < 1 \tag{C3}$$

and

$$\sum_p |\langle ll'p - p \,|\, ll'L0 \rangle| \leqslant \min\left[(2l+1)^{1/2}, (2l'+1)^{1/2}\right] \tag{C4}$$

For the spherical harmonics we use an upper bound (that follows from the addition theorem in a special point):

$$|Y_{lp}(\theta, \phi)| \leqslant \left(\frac{2l+1}{4\pi} \right)^{1/2} \tag{C5}$$

With all these bounds we find from (3.17b)

$$\begin{bmatrix} \frac{N}{2} \\ \frac{n}{2} \end{bmatrix} |\mu_{nll'm}^{NLM}|$$

$$\leqslant \left[\frac{(2l+1)(2l'+1)}{2L+1} \right]^{1/2} \min\left[(2l+1)^{1/2}, (2l'+1)^{1/2}\right] \mu_{N/2,\, n/2} \tag{C6}$$

with $\mu_{N/2,n/2}$ as defined in (3.18b). The triangle inequality $|l - l'| < L < l + l'$ implies

$$\frac{\min\left[(2l+1)^{1/2}, (2l'+1)^{1/2}\right]}{(2l+1)^{1/2}(2l'+1)^{1/2}} \leqslant \left(\frac{2}{2L+1} \right)^{1/2} \tag{C7}$$

Combining (C6) and (C7) the proof of the desired property (C1) is reduced to proving:

$$\max_{N,L} \frac{1}{\lambda_{NL}} \sum_{n=2}^{N-2} \mu_{N/2,n/2} < \infty \tag{C8}$$

which can be done in exactly the same way as in the two-dimensional case (see Appendix B). We define a set of numbers θ_{nlm} by

$$\theta_{nlm} = \frac{\theta_n}{C_1 C_2 C_3 (2l+1)^{\alpha+1} (|m|+1)^{\beta} \Gamma(n/2+1)} \tag{C9}$$

with $\theta_k \theta_{n-k} \leqslant \theta_n$, $\alpha > 2$, $\beta > 1$, C_1 as defined by (C1), C_2 as defined by (2.19), and C_3 defined as

$$C_3 = \max_L \sum_{ll'}{}' \left[\frac{2L+1}{(2l+1)(2l'+1)} \right]^{\alpha} \tag{C10}$$

where the sum is over all values of l and l' compatible with the triangle inequality, and for which $l + l' + L$ is even (for $\alpha > 2$ we have $C_3 < \infty$). The following theorem can now be proved exactly as in the two-dimensional case:

$$|C_{NLM}(0)| < \theta_{NLM} \Rightarrow |C_{NLM}(t)| < \theta_{NLM} \tag{C11}$$

and from this the uniform convergence and convergence in the mean of the series solution (3.7) for the class of initial conditions with $\theta_n = p^n$ ($p \leqslant \pi^{-2/3}$). Here our aim was to prove that a nontrivial class of initial conditions exists, for which the series solution makes sense. We have used rather rough bounds and do not doubt they can be refined, allowing a bigger class of initial conditions. Also exponential bounds of the type considered by Bobylev[17] [$|C_{NLM}(t)| < \theta_{NLM} \exp(-\delta_{NLM} t)$] can be constructed easily.

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